Números complejos

$$x^2+1=0 \quad \rightarrow x=?$$

$$z=(x,y) \rightarrow \text{número complejo}$$

$\boxed{\text{Operaciones}}$: $z_1=(x_1,y_1)$, $z_2=(x_2,y_2)$, $z=(x,y)$

Suma : $z_1+z_2=(x_1+x_2,\ y_1+y_2)$ $\rightarrow$ Neutro : $(0,0)$

Producto : $z_1 \cdot z_2=(x_1 \cdot x_2-y_1 \cdot y_2,\ x_1 y_2+x_2 y_1)$ $\rightarrow$ Neutro : $(1,0)$

$\hookrightarrow$ Inverso : $z \cdot z^{-1}=1 \iff z^{-1}=\left(\frac{x}{x^2+y^2},\ \frac{-y}{x^2+y^2}\right)$

si $z \neq (0,0)$

$\hookrightarrow$ Cociente : $z_1 : z_2 = \frac{z_1}{z_2} = z_1 \cdot z_2^{-1}$

Llamamos : $i=(0,1)$, $1=(1,0)$ $\Rightarrow$ $i^2=i \cdot i=(-1,0)$, $i^2=-1$

$$z=(x,y)=(x,0)+(0,y)=x1+iy$$

$$x \cdot (1,0)+y(0,1)=x+iy$$

$x=\text{Re}(z)$

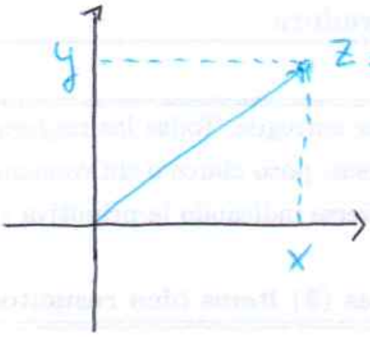
$y=\text{Im}(z)$

Módulo : $|z|=\sqrt{x^2+y^2}$

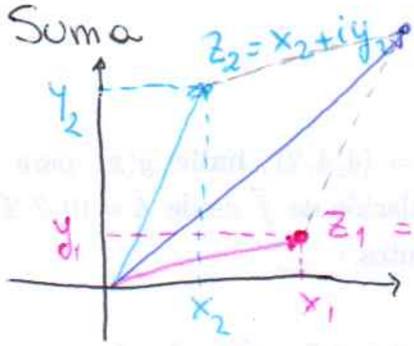
Conjugado : $\bar{z}=x-iy$

$\Rightarrow$ $|z|^2=x^2+y^2=z \cdot \bar{z}$, $|z|=|\bar{z}|$

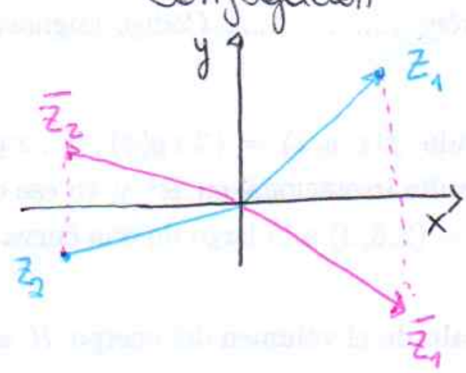
Representación geométrica



Suma



Conjugación



Forma polar

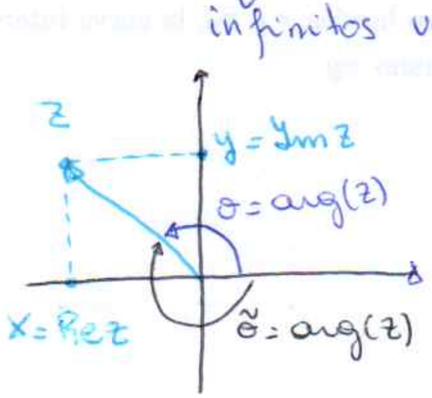
(r,θ) : coordenadas polares de $(x,y) \neq (0,0)$

$z = x+iy$

$$\boxed{r = |z| = \sqrt{x^2+y^2}}$$

$\boxed{\theta = \arg(z)}$ tal que $\frac{x}{|z|} = \frac{\operatorname{Re} z}{|z|} = \cos\theta$ y $\frac{y}{|z|} = \frac{\operatorname{Im}(z)}{|z|} = \operatorname{sen}\theta$

infinitos valores!



$\theta - \tilde{\theta} = 2\pi$

Argumento principal

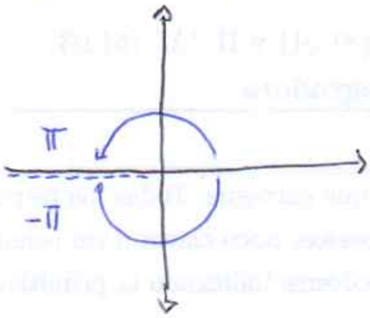
$\Theta = \operatorname{Arg}(z)$ con $-\pi < \operatorname{Arg}(z) \leq \pi$

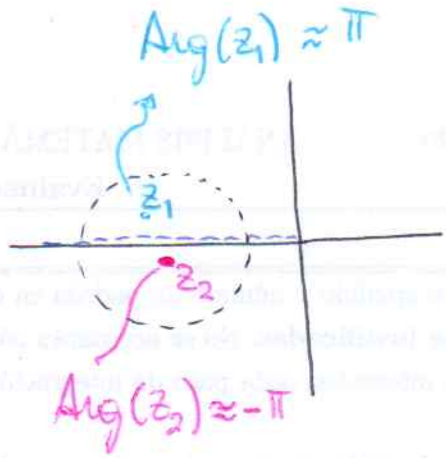
Luego: $\arg(z) = \operatorname{Arg}(z) + 2k\pi, \ k \in \mathbb{Z}$

$z = x+iy = r\cos\theta + i r \operatorname{sen}\theta$

$$\boxed{z = r(\cos\theta + i\operatorname{sen}\theta)}$$ $\rightarrow$ si $z \neq 0$

Argumento principal





Propiedades

* $|z_1 \cdot z_2| = |z_1| \cdot |z_2| \longrightarrow |z^2| = |z|^2 \; ; \; |z^n| = |z|^n$
* $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
* $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
* $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

$\rightarrow |z^{-1}| = \dfrac{1}{|z|}$

$\rightarrow \arg(z^{-1}) = -\arg(z)$

Forma exponencial

$e^{i\theta} = \cos\theta + i \operatorname{sen}\theta \longrightarrow |e^{i\theta}| = 1$

$(e^{i\theta})^{-1} = e^{i(-\theta)}$

$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

$e^{i\theta} = e^{i(\theta+2k\pi)} \quad k \in \mathbb{Z}$

$\boxed{z = re^{i\theta}}$ $\quad r = |z| \quad z \neq 0$

$\theta = \arg(z)$

Potencias y raíces

$$z = x+iy \longrightarrow z^n = (x+iy)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (iy)^k$$

Mejar...

$$z = re^{i\theta} \longrightarrow z^n = r^n e^{in\theta} \quad n \in \mathbb{Z}$$

Raíces $\;¿\sqrt[n]{z}\,?$ Se busca ω tal que $\omega^n = z$

Si $z = re^{i\theta}$ $\quad \omega = \rho e^{i\alpha} \Rightarrow \omega^n = \rho^n e^{in\alpha}$

$$\omega^n = z$$

$$\rho^n e^{in\alpha} = re^{i\theta}$$

$$\begin{cases} \rho^n = r \\ n\alpha = \theta + 2k\pi \quad k \in \mathbb{Z} \end{cases}$$

$$\boxed{\begin{aligned} \rho &= \sqrt[n]{r} \\ \alpha &= \frac{\theta}{n} + \frac{2k\pi}{n} \quad k \in \mathbb{Z} \end{aligned}}$$

con $k=0$ y $k=n$ se obtiene el mismo argumento.

Ejemplo: $z^3 + i = 0$

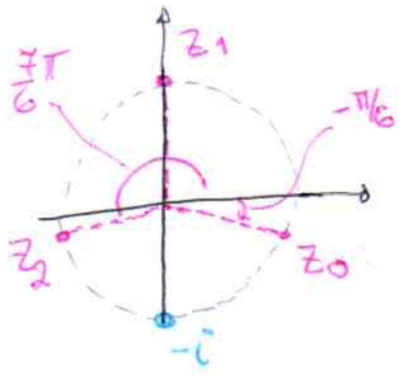
$$z^3 = -i$$

En forma exponencial: $z = \rho e^{i\alpha} \longrightarrow z^3 = \rho^3 e^{i3\alpha}$

$$-i = 1 \cdot e^{i\left(-\frac{\pi}{2}\right)}$$

$$\rho^3 e^{i3\alpha} = 1 e^{i\left(-\frac{\pi}{2}\right)}$$

$$\begin{cases} \rho^3 = 1 \\ 3\alpha = -\frac{\pi}{2} + 2k\pi \end{cases} \longrightarrow \begin{cases} \rho = 1 \\ \alpha = -\frac{\pi}{6} + \frac{2k\pi}{3} \quad k \in \mathbb{Z} \end{cases}$$



$k=0 \rightarrow z_0 = e^{-\pi/6\, i}$

$k=1 \rightarrow z_1 = e^{\pi/2\, i}$

$k=2 \rightarrow z_2 = e^{7/6\,\pi i}$

Funciones complejas

$$z \to \boxed{f} \to w = f(z)$$

$$w = u + iv$$

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

o:

$$f(z) = f(re^{i\theta}) = U(r,\theta) + i\,V(r,\theta)$$

<u>Ejemplo</u>:

- $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2-y^2}_{u(x,y)=x^2-y^2} + i\,\overbrace{2xy}^{v(x,y)=2xy}$

- $f(z) = \dfrac{1}{z} = \dfrac{\bar{z}}{z\cdot\bar{z}} = \dfrac{\bar{z}}{|z|^2} = \underbrace{\dfrac{x}{x^2+y^2}}_{u(x,y)=\frac{x}{x^2+y^2}} + i\,\underbrace{\dfrac{-y}{x^2+y^2}}_{v(x,y)=\frac{-y}{x^2+y^2}}$

- $f(z) = \underbrace{x^2+y}_{u(x,y)=x^2+y^2} + i\,\underbrace{\cos(x)\cdot y}_{v(x,y)=\cos(x)\cdot y}$

- $f(z) = z^2 = (re^{i\theta})^2 = r^2e^{i2\theta} = \underbrace{r^2\cos(2\theta)}_{U(r,\theta)=r^2\cos(2\theta)} + i\,\underbrace{r^2\operatorname{sen}(2\theta)}_{V(r,\theta)=r^2\operatorname{sen}(2\theta)}$

$\boxed{\text{Límite}}$ $f: D \subset \mathbb{C} \to \mathbb{C}$, z_0: punto de acumulación de D.

$\lim_{z\to z_0} f(z) = l \iff$ para cada $\varepsilon > 0$, existe $\delta > 0$ tal que:

$$\begin{matrix} 0 < |z-z_0| < \delta \\ z \in D \end{matrix} \Rightarrow |f(z)-l| < \varepsilon$$

Propiedades

* $\lim_{z \to z_0} f(z) = l \iff \lim_{z \to z_0} |f(z) - l| = 0$

* álgebra de límites ...

* $f(z) = u(x,y) + i\,v(x,y)$
  $z_0 = x_0 + i y_0$

$$\lim_{z \to z_0} f(z) = a + ib \iff \begin{cases} -\lim_{(x,y) \to (x_0,y_0)} u(x,y) = a \\ -\lim_{(x,y) \to (x_0,y_0)} v(x,y) = b \end{cases}$$

Continuidad

$f : D \subset \mathbb{C} \to \mathbb{C}$, $z_0 \in D$.

f es continua en z_0 si para cada $\varepsilon > 0$, existe $\delta > 0$ tal que

$$\underset{z \in D}{|z - z_0| < \delta} \implies |f(z) - f(z_0)| < \varepsilon$$

Nota: si $z_0 \in D$ es punto de acumulación de D:

f es continua en $z_0 \iff \lim_{z \to z_0} f(z) = f(z_0)$

Límites infinitos

- $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$

- $\lim_{z \to \infty} f(z) = l \iff \lim_{w \to 0} f\left(\frac{1}{w}\right) = l$